§4.5 Dimension of a Vector Space

We generalize the ideas from §2.9 to any vector space (not just $\mathbb{R}^n$).

Theorem

If a vector space V has a basis consisting of n vectors, then every basis of V has n vectors.

In other words, the number of basis vectors is unique

Definition

Let V be a vector space. The dimension of V is the number of vectors in any basis of V. We denote it $\dim V$.

- If V has a basis with finitely many vectors, we say V is finite-dimensional
- If V does not have a finite basis, i.e. is not spanned by a finite set, then V is infinite-dimensional.

Examples

- $\dim \mathbb{R}^n = n$ since $\{e_1, \ldots, e_n\}$ is a basis

- $\dim \mathbb{P}_n = n+1$ since $\{1, t, t^2, \ldots, t^n\}$ is a basis

- $\dim \{0\} = 0$ and this is the only zero-dimensional space.

- $\dim M_{m \times n} = m \cdot n$ exercise: come up with a basis!

- $\mathbb{P}$ is the vector space of <u>all</u> polynomials. (any degree!)

  $\dim \mathbb{P} = \infty$ since it has infinite basis

  $$\{1, t, t^2, \ldots\ldots\}$$

- If V is the space of real-valued continuous functions, then $\dim V = \infty$

  Makes sense since $\mathbb{P} \subset V$.

Recall that a basis $\mathcal{B}$ for a vector space V is both

1) The smallest spanning set
2) The largest linearly independent set

which leads to the following result.

Theorem

Let V be a nonzero finite-dimensional vector space and $\{u_1, \ldots, u_\ell\}$ a set of vectors in V.

a) If $\ell < \dim V$, then $\{u_1, \ldots, u_\ell\}$ does not span V. (It does span a proper subspace however).

b) If $\ell > \dim V$, then $\{u_1, \ldots, u_\ell\}$ is not a linearly independent set.

Theorem

Let H be a subspace of finite-dimensionl vector space V. Any basis of H can be extended to a basis of V. Moreover

$$\dim H \leq \dim V$$

Theorem (Basis Theorem)

Let V be a vector space with $\dim V = n$ (finite)

a) Any set of n linearly independent vectors is a basis of V.

b) Any set of n vectors that spans V is a basis of V.

Example

Find a basis and the dimension of the following subspace of $\mathbb{R}^4$

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \middle| \; x - 4y + w = 0 \right\}$$

notice $x = 4y - w$ so

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4y - w \\ y \\ z \\ w \end{bmatrix} = y\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

since we can't eliminate any more variables

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for H

so $\dim H = 3$

This is just one basis! We chose to let $x = 4y - w$, but we could have taken $w = 4y - x$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 4y-x \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is also a basis.

Lots of choices, but they all have the same number of vectors!

Let A be a matrix. Recall:

- The <u>rank</u> of A is the dimension of the column space, $\text{rank } A = \dim \text{Col } A$
- The <u>nullity</u> of A is the dimension of the null space, ~~dim~~ $\text{nullity } A = \dim \text{Nul } A$.

We had seen the rank theorem before in section §2.9

<u>Theorem (Rank Theorem)</u>

$$\underset{\substack{\shortparallel \\ \#\text{ pivot columns}}}{\text{rank } A} + \underset{\substack{\shortparallel \\ \#\text{ non-pivot columns}}}{\dim \text{Nul } A} = \#\text{ columns of } A$$

If A is $m \times n$, we can see these spaces in the matrix transformation given by A

$$T: \underset{\substack{\cup\!| \\ \text{Nul } A}}{\mathbb{R}^n} \longrightarrow \underset{\substack{\cup\!| \\ \text{Col } A}}{\mathbb{R}^m} \quad \text{by} \quad T(x) = Ax$$

$$\text{rank } A + \dim \text{Nul } A = n \quad (\#\text{ columns here})$$

We can generalize this result to any vector space!

Theorem

Let V and W be vector spaces and let $T: V \longrightarrow W$ be a linear transformation.

Then

$$\dim \ker T + \dim \operatorname{range} T = \dim V$$

Example

Recall from lesson on 3/10

$T: \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ by $T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$ has

$\ker T = \operatorname{span}\{t^2 - t\}$

$\operatorname{range} T = \mathbb{R}^2$

$$\begin{array}{r} \dim \ker T = 1 \\ + \dim \operatorname{range} T = 2 \\ \hline 3 \end{array}$$

and we know $\dim \mathbb{P}_2 = 3$